References


Mathematical Models for Solving Arithmetic Problems

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Introduction

One way to enhance students' ability in solving problems is to help them visualise abstract mathematical relationships and various problem structures through pictorial representations. In this article, the use of mathematical models in the form of pictorial representations is discussed. Examples are given to show how the models can be used to solve various problems. The main features are:

(a) The model method helps students gain a better insight into mathematical concepts such as fraction, ratio and percentage.

(b) The model method helps students plan the solution steps for solving an arithmetic word problem.

(c) The model method is comparable to, but is less abstract than the algebraic method.

(d) The model method can motivate students to solve challenging problems.

Mathematical Models

The Part-Whole Model

The part-whole model (also known as the 'part-part-whole' model) shows the various parts which make up a whole, e.g.

\[
\begin{array}{c}
\text{whole} \\
\text{part} \\
\text{part}
\end{array}
\]

Here the whole is divided into two parts. When the two parts are given, we can find the whole by addition. When the whole and one part are known, we can find the other part by subtraction.
Example 1

Minah goes shopping. She buys a handbag for $120 and is left with $168. How much money does she have at first?

We draw a bar to represent the amount of money Minah has at first. It is marked with a question mark (?) to indicate that the amount is unknown. As shown in the model, both parts are known, so we find the whole by addition:

$120 + $168 = $288

Minah has $288 at first.

In the part-whole model, the whole may be divided into more than two parts. We can find the whole by multiplying one part by the number of parts, when all the parts are equal, e.g.

Conversely, given the whole, we can find one part or the number of parts by division.

The Comparison Model

The comparison model shows the relationship between two quantities when they are compared. We may compare two quantities by showing their difference, e.g.

A is 20 more than B; or B is 20 less than A. In other words, the difference between A and B is 20.
We may also compare two quantities by showing their ratio, e.g.

\[ C \]
\[ D \]

\( C \) is \( \frac{3}{2} \) of \( D \); or \( D \) is \( \frac{2}{3} \) of \( C \). In other words, the ratio of \( C \) to \( D \) is \( 2 : 3 \).

When two quantities are given, we can find the difference or ratio. Conversely, when one quantity and the difference or ratio are given, we can find the other quantity.

**The Change Model**

The change model shows the relationship between the new value of a quantity and its original value after an increase or a decrease, e.g.

![New Value Diagram](new-value-diagram)

The increase is 12.

The increase or decrease may be expressed as a fraction of the original value, e.g.

![Increase Fraction Diagram](increase-fraction-diagram)

The increase is \( \frac{2}{5} \) of the original value.

The increase or decrease may also be expressed as a percentage of the original value, e.g.

![Increase Percentage Diagram](increase-percentage-diagram)

The decrease is 30% of the original value.

Knowing the increase or decrease, we can find the new value from the original value and vice-versa.
Problems Involving Fraction, Ratio and Percentage

The use of models will help students understand and solve arithmetic word problems. When dealing with problems involving fraction, ratio and percentage, it will also help students gain a better insight into the various mathematical concepts.

In the following examples, the model method as well as the usual method for solving problems are presented. The usual method is abstract and if this is the only method taught, many students will have difficulty in solving problems. They will resort to learning by rote. On the other hand, the model method makes use of a pictorial model to illustrate the concept of fraction, ratio and percentage. The solution steps are explicit. Therefore if the model method is taught prior to the usual method, it will help students understand the usual method.

Example 2

Minah had 20 m of cloth. She used $\frac{3}{5}$ of it to make some dresses for her dolls. How many metres of cloth did she use?

The model method

![Diagram showing 20 m of cloth divided into 5 equal parts, with 3 parts shaded]

We use the part-whole model to show $\frac{3}{5}$ of 20 m. Here the whole is divided into 5 equal parts of which 3 parts are shaded. In other words, the whole comprises 5 units, so $\frac{3}{5}$ of it comprises 3 units. We have:

- 5 units = 20 m
- 1 unit = $20 \div 5 = 4$ m
- 3 units = $3 \times 4 = 12$ m

She used 12 m of cloth.

The usual method

Amount of cloth used = $\frac{3}{5} \times 20 = 12$ m
Example 3

Devi and Minah shared a prize money of $420 in the ratio 4 : 3. How much money did Devi receive?

The model method

Suppose the whole comprises 7 units, then the ratio 4 : 3 means Devi received 4 units and Minah received 3 units. We have:

7 units = $420
1 unit = $420 ÷ 7 = $60

Devi’s share = 4 units = 4 × $60 = $240

The usual method

Devi’s share = \( \frac{4}{7} \times $420 = $240 \)

Example 4

A book contains 200 pages. 35% of the pages are in colour. How many pages are in colour?

The model method

Suppose the whole comprises 100 units, then 35% of it comprises 35 units. We have:

100 units = 200
1 unit = 200 ÷ 100 = 2
35 units = 35 × 2 = 70

70 pages are in colour.
The usual method

Number of pages in colour = 35% \times 200 = \frac{35}{100} \times 200 = 70

The Model Method and the Algebraic Method

Without the models, students may have to resort to the algebraic method to solve structurally complex problems. The model method is less abstract than the algebraic method and can be introduced before students learn to solve algebraic equations. Indeed the models serve as good pictorial representations of algebraic equations. If students are taught the model method first, when they eventually learn the algebraic method, they will appreciate better the use of symbols to represent quantities as they have experienced using bars to represent quantities in the model method.

Example 5

Devi is 10 kg heavier than Minah. Their total mass is 100 kg. Find Devi’s mass.

The model method

We use the comparison model to show the relationship between Devi’s mass and Minah’s mass. Suppose Minah’s mass is 1 unit, then Devi’s mass is 1 unit and 10 kg as shown.

The total mass is 2 units + 10 kg. We have:

\[ \begin{align*}
2 \text{ units} &= 100 - 10 = 90 \text{ kg} \\
1 \text{ unit} &= 90 \div 2 = 45 \text{ kg} \\
\text{Devi’s mass} &= 45 + 10 = 55 \text{ kg}
\end{align*} \]
The algebraic method

Let Minah’s mass be \( x \) kg, then Devi’s mass is \((x + 10)\) kg. We have:

\[
\text{Total mass} = x + (x + 10) = (2x + 10) \text{ kg}
\]

Hence, \(2x + 10 = 100\)
\[
2x = 90
\]
\[
x = 45
\]

Devi’s mass = \(x + 10 = 45 + 10 = 55\) kg

**Challenging Problems**

The following examples show how the model method can motivate students to solve challenging problems in an elementary and interesting manner. The method has been established as a generalised method for solving arithmetic word problems. It is a synthetic-analytic process. First, we construct a model to help us describe and interpret the problem situation and understand the problem structure by processing the given information (the synthetic approach). Then we use the model to help us develop a sequence of logical steps for the solution of the problem (the analytic approach). The model reveals the hidden information and helps us relate unknown and known quantities. It is a powerful tool for solving complex problems.

**Example 6**

*Devi and Minah have $520 altogether. If Devi spends \(\frac{2}{5}\) of her money and Minah spends $40, then they will have the same amount of money left. How much money does Devi have?*

![Diagram showing the comparison model in combination with the change model. The shaded parts show that the two remaining amounts of money are equal (i.e. 3 units each). The total amount of money is 8 units + $40. We have:]

Here we use the comparison model in combination with the change model. The shaded parts show that the two remaining amounts of money are equal (i.e. 3 units each). The total amount of money is 8 units + $40. We have:

\[
8 \text{ units} = 520 - 40 = 480
\]
\[
1 \text{ unit} = 480 \div 8 = 60
\]

Devi’s money = 5 units = 5 \times 60 = $300
Example 7

Devi’s salary and Minah’s salary are in the ratio 4 : 5. If Devi’s salary is increased by 30%, by what percentage must Minah’s salary be increased or decreased so that they will have the same salary?

The model shows that Devi’s salary is 4 units and Minah’s salary is 5 units at first. It also shows that their new salaries are equal. We have:

- Increase in Devi’s salary = 30% × 4 units = 1.2 units
- Increase in Minah’s salary = 1.2 units – 1 unit = 0.2 unit
- Percentage increase = \( \frac{0.2}{5} \times 100\% = 4\% \)

Conclusion

The model method was introduced in Singapore at the Primary Four level in 1983. Teachers are encouraged to use the method in their classrooms. Generally, the method is well received, and teachers are able to assign structurally complex problems as challenging problems to the more able students. By doing so, the students will be more receptive to unfamiliar problems and their problem-solving ability will be enhanced. However, it will take several years for all teachers to be acquainted with this method and teach it in their classrooms. In the meantime, some studies can be carried out in the classrooms to determine the effectiveness of the method and to improve the instruction of the method.